

Close Friday: 15.3

Close next Thurs: 15.4

**Note:** I won't ask about 15.4 (center of mass) on exam 2, but use the 15.4 HW for more practice with double integrals.

### Exam 2, Tuesday, May 15<sup>th</sup>

13.4: Acceleration/Velocity/Dist

### 14.1/3/4/7: Partial derivatives

Level Curves, Domain, Partials, Tangent Plane, local max/min (2<sup>nd</sup> deriv. test), global max/min, applied max/min

### 15.1-15.3: Double integrals

general regions (top/bot, left/right), reversing order, polar

**Entry Task:** How do you start these?

**HW 15.4:** Find the volume enclosed by

$$-x^2 - y^2 + z^2 = 22 \text{ and } z = 5.$$

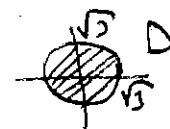
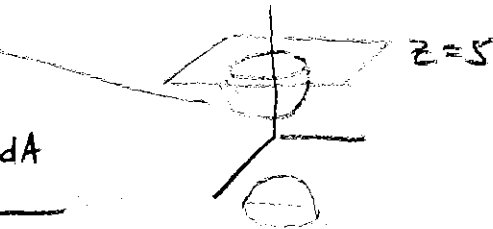
$$z^2 = 22 + x^2 + y^2$$

$$z = \pm \sqrt{22 + x^2 + y^2}$$

$$\iint_D \sqrt{22 + x^2 + y^2} \, dA - \iint_D \sqrt{22 + x^2 + y^2} \, dA$$

INTERSECT  $25 = 22 + x^2 + y^2$

$$3 = x^2 + y^2$$



**HW 15.4:** Find the volume above the

upper cone  $z = \sqrt{x^2 + y^2}$  and below

$$x^2 + y^2 + z^2 = 81$$

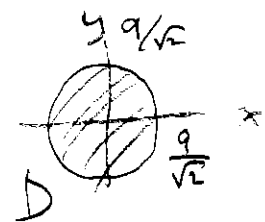
$$\iint_D \sqrt{81 - x^2 - y^2} \, dA - \iint_D \sqrt{x^2 + y^2} \, dA$$

$$x^2 + y^2 + \sqrt{x^2 + y^2} = 81$$

$$\Rightarrow x^2 + y^2 + x^2 + y^2 = 81$$

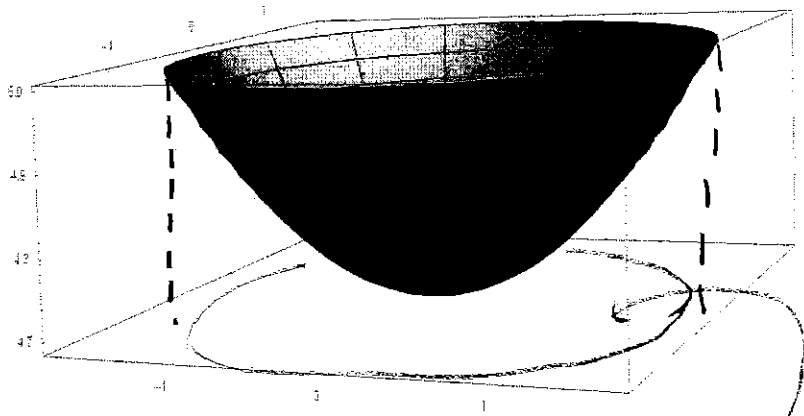
$$\Rightarrow 2x^2 + 2y^2 = 81$$

$$\Rightarrow x^2 + y^2 = \frac{81}{2}$$



Volume enclosed by

$$-x^2 - y^2 + z^2 = 22 \text{ and } z = 5.$$

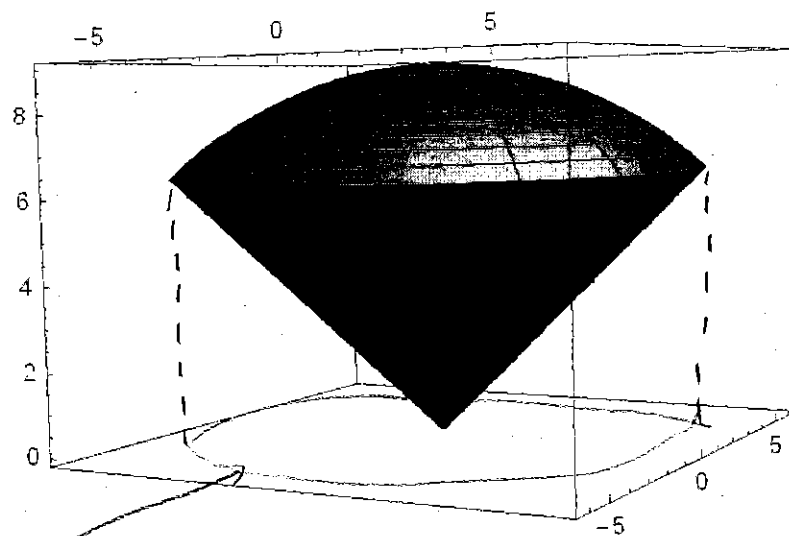


ALL THE VOLUME IS ABOVE THIS  
CIRCULAR REGION

The volume above the upper cone

$$z = \sqrt{x^2 + y^2} \text{ and below}$$

$$x^2 + y^2 + z^2 = 81$$



ALL THE VOLUME IS ABOVE  
THIS CIRCULAR REGION.

Close Thu:

15.3

Exam 2, Tues, May 15<sup>th</sup> (13.4,14.1/3/4/7,15.1-4)

### 15.3 Double Integrals over Polar Regions

Recall:

$\theta$  = angle measured from positive x-axis

$r$  = distance from origin

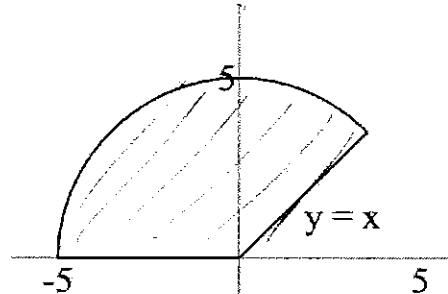
$$x = r \cos(\theta), y = r \sin(\theta), x^2 + y^2 = r^2$$

To set up a double integral in polar we:

1. Describing the region in polar
2. Replace "x" by " $r \cos(\theta)$ "
3. Replace "y" by " $r \sin(\theta)$ "
4. Replace "dA" by " $r dr d\theta$ "

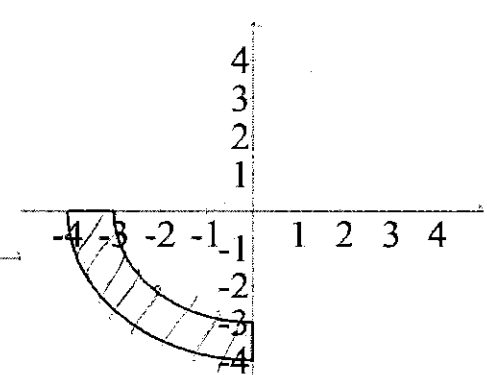
### Step 1: Describing regions in polar.

Examples: Describe the regions



$$\frac{\pi}{4} \leq \theta \leq \pi$$

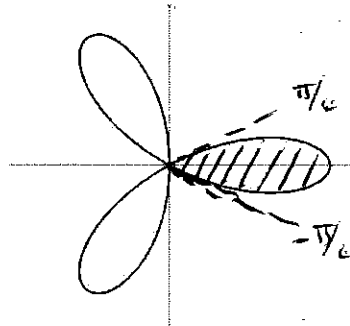
$$0 \leq r \leq 5$$



$$\pi \leq \theta \leq \frac{3\pi}{2}$$

$$3 \leq r \leq 4$$

### HW 15.3: One loop of $r = 6 \cos(3\theta)$ .



$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$

$$0 \leq r \leq 6 \cos(3\theta)$$

FIND WHEN  $r = 0$

$$6 \cos(3\theta) = 0$$

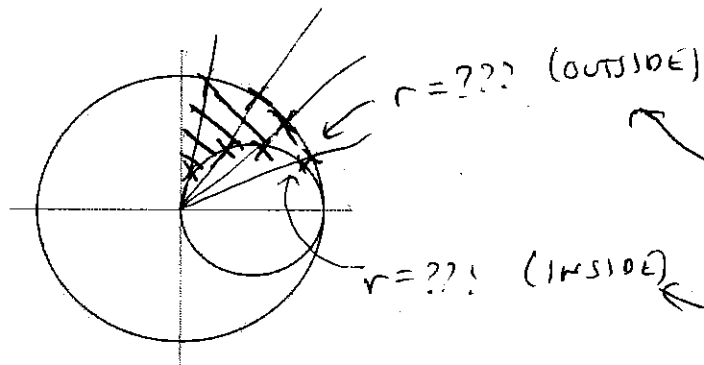
$\Rightarrow$

$$3\theta = \dots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\Rightarrow \theta = \dots, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \dots$$

CONSECUTIVE

**HW 15.4:** Region in the first quadrant between the circles  $x^2 + y^2 = 16$  and  $x^2 + y^2 = 4x$ .



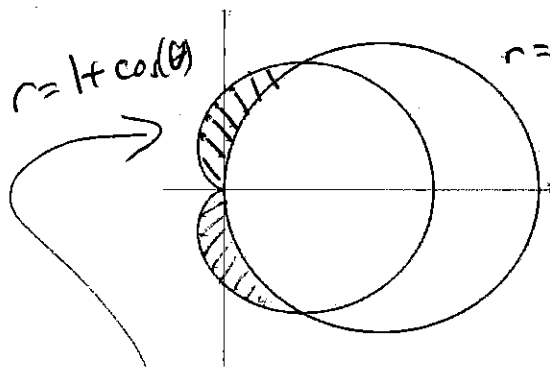
$$x^2 + y^2 = 16 \Leftrightarrow r^2 = 16 \Leftrightarrow r = 4$$

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r \cos \theta \Leftrightarrow r = 4 \cos \theta$$

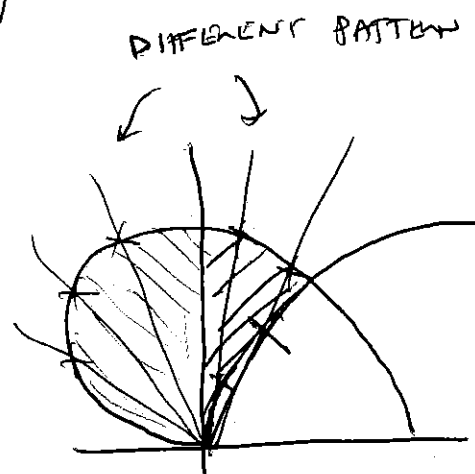
$$0 \leq \theta \leq \frac{\pi}{2}$$

$$4 \cos \theta \leq r \leq 4$$

**HW 15.4:** Describe the region inside  $r = 1 + \cos(\theta)$  and outside  $r = 3 \cos(\theta)$ .



$r = 3 \cos \theta$  TRIGKY



INTERSECT?  $1 + \cos \theta = 3 \cos \theta \Leftrightarrow 1 = 2 \cos \theta$   
 $\Leftrightarrow \frac{1}{2} = \cos \theta$   
 $\Leftrightarrow \theta = \frac{\pi}{3}$

$$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$$

I

 $3 \cos \theta \leq r \leq 1 + \cos \theta$

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II

 $0 \leq r \leq 1 + \cos \theta$

SIMILAR  
FOR LOWER  
PART

## Step 2: Set up your integral in polar.

Conceptual notes:

*Cartesian*

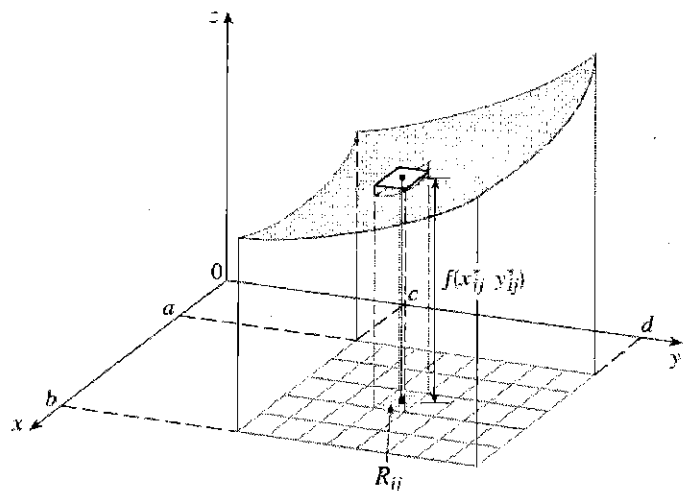
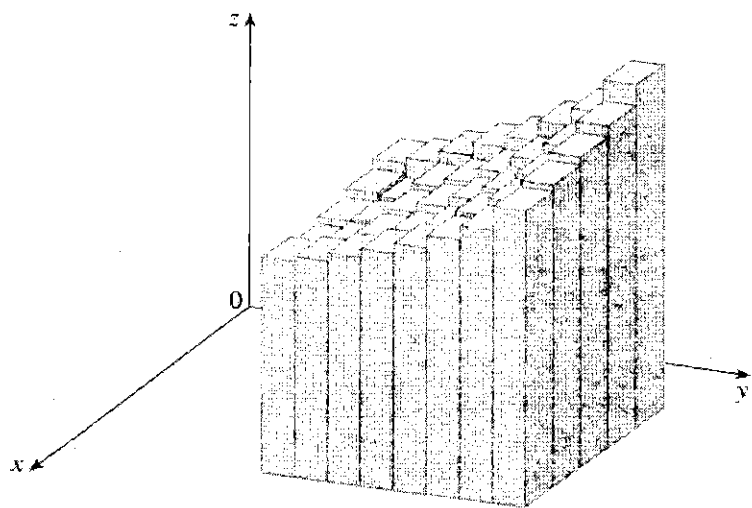
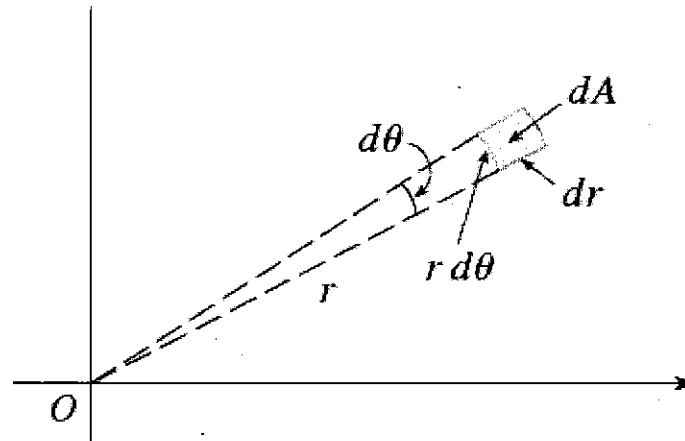
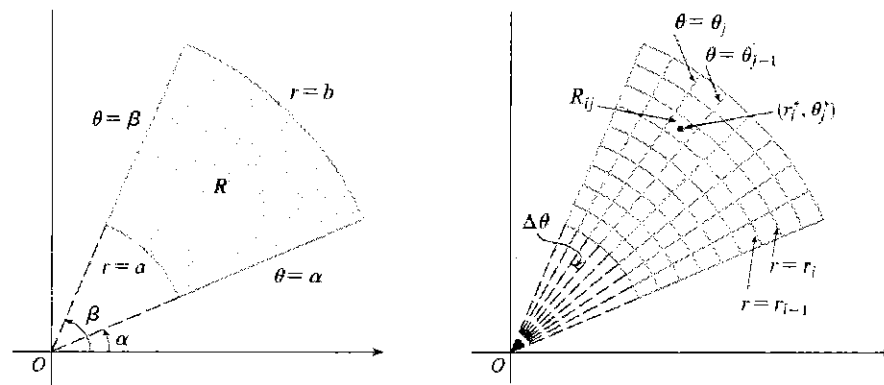


FIGURE 4



*Polar*



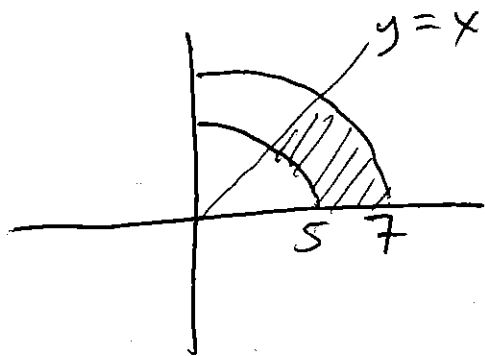
Examples:

1. Compute

$$\iint_R \frac{\cos(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} dA$$

where R is the region in the first quadrant that is between  $x^2 + y^2 = 49$ ,  $x^2 + y^2 = 25$  and below  $y = x$ .

**STEP 1** DRAW REGION  
GET BOUNDS



$$0 \leq \theta \leq \frac{\pi}{4}$$
$$5 \leq r \leq 7$$

**STEP 2**

$$\int_0^{\pi/4} \left( \int_5^7 \frac{\cos(r)}{r} r dr d\theta \right)$$

$$\int_0^{\pi/4} \sin(r) \Big|_5^7 d\theta$$

$$\int_0^{\pi/4} (\sin(7) - \sin(5)) d\theta$$

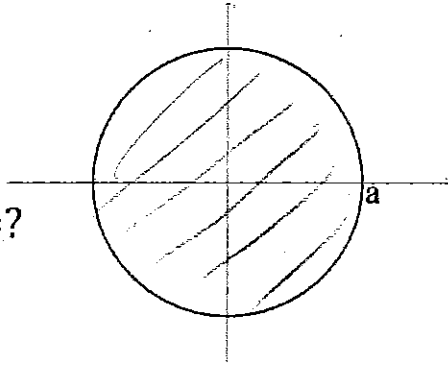
$$(\sin(7) - \sin(5)) \theta \Big|_0^{\pi/4}$$

$$(\sin(7) - \sin(5)) \frac{\pi}{4}$$

2. Set up the two double integrals below over the entire circular disc of radius  $a$ :

$$\iint_D 1 \, dA = ?$$

$$\iint_D \sqrt{a^2 - x^2 - y^2} \, dA = ?$$



$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq a$$

$$\iint_D 1 \, dA = \int_0^{2\pi} \int_0^a 1 \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left. \frac{1}{2} r^2 \right|_0^a \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} a^2 \, d\theta$$

$$= \left. \frac{1}{2} a^2 \theta \right|_0^{2\pi}$$

$$= \frac{1}{2} a^2 2\pi$$

$$= \boxed{\pi a^2}$$

$$= \pi (\text{RADIUS})^2 = \text{AREA OF CIRCLE!}$$

$$\iint_D \sqrt{a^2 - x^2 - y^2} \, dA$$

$$= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} \, r \, dr \, d\theta$$

$$u = a^2 - r^2$$

$$du = -2r \, dr$$

$$\frac{1}{-2r} du = dr$$

$$= \int_0^{2\pi} \left[ \int_{a^2}^0 \sqrt{u} \cdot \frac{1}{-2r} du \right] d\theta$$

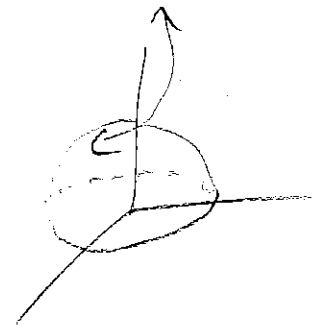
$$= \int_0^{2\pi} \left. -\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \right|_{a^2}^0 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} a^3 \, d\theta$$

$$= \left. \frac{1}{3} a^3 \theta \right|_0^{2\pi}$$

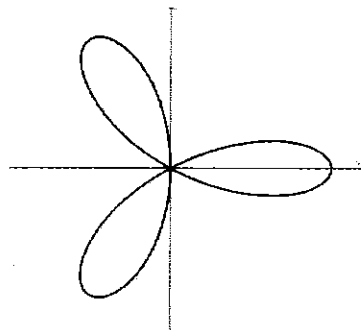
$$= \boxed{\frac{2}{3} \pi a^3}$$

HALF OF VOLUME  
OF A SPHERE



### 3. HW 15.3:

Find the area of one closed loop of  
 $r = 6\cos(3\theta)$ .



**RECALL**

$$\iint_D 1 \, dA = \text{"AREA OF } D \text{"}$$

EARLIER WE FOUND

$$-\pi/6 \leq \theta \leq \pi/6$$

$$0 \leq r \leq 6\cos(3\theta)$$

$$\int_{-\pi/6}^{\pi/6} \int_0^{6\cos(3\theta)} 1 \cdot r \, dr \, d\theta$$

$$\int_{-\pi/6}^{\pi/6} \frac{1}{2} r^2 \Big|_0^{6\cos(3\theta)} d\theta$$

$$\int_{-\pi/6}^{\pi/6} \frac{1}{2} 36 \cos^2(3\theta) d\theta$$

↓ HALF-ANGLE!!!

$$18 \int_{-\pi/6}^{\pi/6} \frac{1}{2} (1 + \cos(6\theta)) d\theta$$

$$9 \left[ \theta + \frac{1}{6} \sin(6\theta) \Big|_{-\pi/6}^{\pi/6} \right]$$

$$9 \left[ \left( \frac{\pi}{6} + 0 \right) - \left( -\frac{\pi}{6} + 0 \right) \right]$$

$$9 - \frac{2\pi}{6} = \text{~~9~~} = \boxed{3\pi}$$



### 3. HW 15.3:

Evaluate

$$\iint_R x \, dA$$

over the region in the first quadrant  
between the circles  $x^2 + y^2 = 16$  and  $x^2 + y^2 = 4x$  using polar

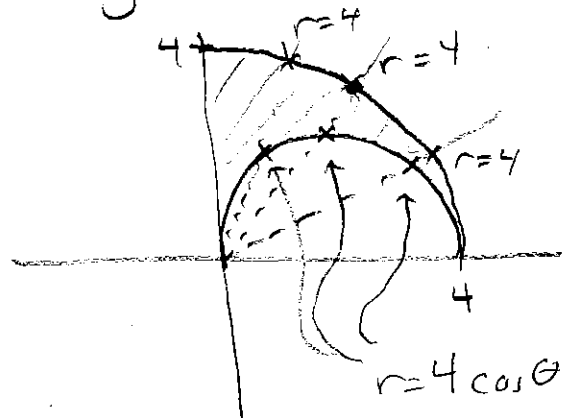
$$0 \leq \theta \leq \frac{\pi}{2}$$

$$4 \cos \theta \leq r \leq 4$$

$$\begin{aligned} & \int_0^{\pi/2} \int_{4 \cos \theta}^4 r \cos \theta \, r \, dr \, d\theta \\ &= \int_0^{\pi/2} \cos \theta \left. \frac{1}{3} r^3 \right|_{4 \cos \theta}^4 d\theta \\ &= \frac{4^3}{3} \int_0^{\pi/2} \cos \theta - \cos^4 \theta \, d\theta \\ &= \frac{64}{3} \left[ \sin \theta \Big|_0^{\pi/2} - \int_0^{\pi/2} \cos^4(\theta) \, d\theta \right] \\ &= \frac{64}{3} - \frac{64}{3} \int_0^{\pi/2} \cos^2(\theta) \cos^2(\theta) \, d\theta \end{aligned}$$

$$x^2 + y^2 = 16 \Leftrightarrow r^2 = 16 \Rightarrow r = 4$$

$$x^2 + y^2 = 4x \Leftrightarrow r^2 = 4r \cos \theta \Rightarrow r = 4 \cos \theta$$



HALF ANGLE

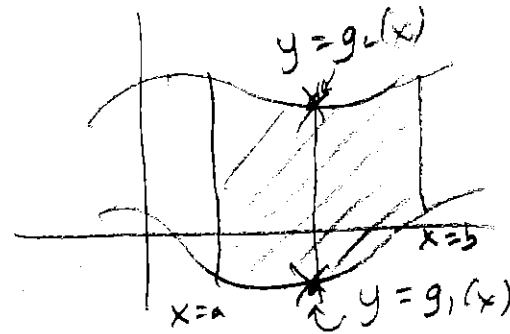
$$\begin{aligned} & \frac{64}{3} - \frac{64}{3} \int_0^{\pi/2} \frac{1}{2} (1 + \cos(2\theta)) \frac{1}{2} (1 + \cos(2\theta)) \, d\theta \\ & \frac{64}{3} - \frac{16}{3} \int_0^{\pi/2} 1 + 2 \cos(2\theta) + \cos^2(2\theta) \, d\theta \\ & \frac{64}{3} - \frac{16}{3} \int_0^{\pi/2} 1 + 2 \cos(2\theta) + \frac{1}{2} + \frac{1}{2} \cos(4\theta) \, d\theta \\ & \frac{64}{3} - \frac{16}{3} \left[ \frac{1}{2} \theta + \sin(2\theta) + \frac{1}{8} \sin(4\theta) \Big|_0^{\pi/2} \right] \\ & \frac{64}{3} - \frac{16}{3} \left[ \left( \frac{3\pi}{2} + 0 + 0 \right) - (0 + 0 + 0) \right] \\ & \boxed{\frac{64}{3} - 4\pi} \end{aligned}$$

## Moral:

Three ways to describe a region in a double integral:

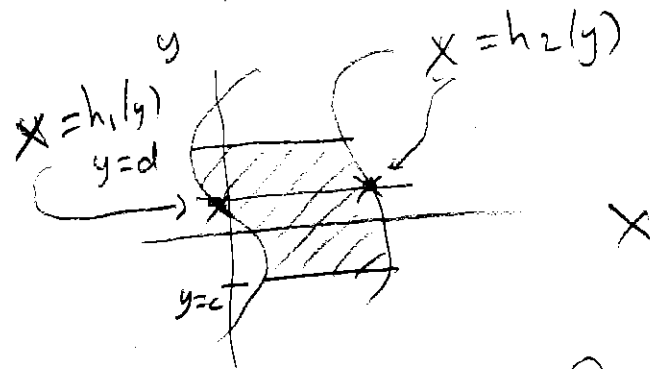
“Top/Bottom”:

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



“Left/Right”:

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$



“Inside/Outside”:

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos(\theta), r \sin(\theta)) r dr d\theta$$

